Indian Statistical Institute

Midterm Exam. 2024-2025

Analysis II, B.Math First Year

Time : 3 Hours   Date : 17.02.2025   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) You may freely apply any of the theorems discussed in class.

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(1) (10 marks) Let $f : [0, \infty) \to [0, \infty)$ be a function. Suppose $\int_0^\infty f$ converges and $\lim_{x\to\infty} f(x)$ exists. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

(2) (20 marks) Prove that if f is a monotone function on $[a, b]$, then f is Riemann-integrable on $[a, b]$.

(3) (20 marks) Consider the greatest integer function $f : [0, 2] \to \mathbb{R}$. Prove that f is Riemann integrable on $[0, 2]$, and compute $\int_0^2 f$.

(4) (20 marks) Let f and g be continuous and real-valued functions defined on $[a, b]$ and suppose $\int_a^b f = \int_a^b g$. Prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

(5) (20 marks) Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [n, n + \frac{1}{2^n}) \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $f|_{[0,r]} : [0, r] \to \mathbb{R}$ is Riemann integrable for all $r > 0$. (i) Prove that $\int_0^\infty f$ converges. (ii) Compute the value of $\int_0^\infty f$. (iii) Prove that $\lim\limits_{x\to\infty} f(x)$ does not exist.

(6) (20 marks) Let f and g be continuous functions defined on $[0, 1]$. Assume that f is monotonically increasing and g is monotonically decreasing. Prove that

$$\int_0^1 fg \leq \Big(\int_0^1 f\Big)\Big(\int_0^1 g\Big).$$